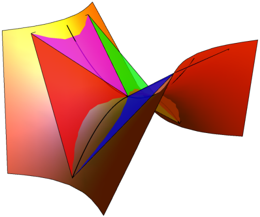


Improving Optimal Triangulation of Saddle Surfaces

Computational Geometric Learning supported by FET-Open grant

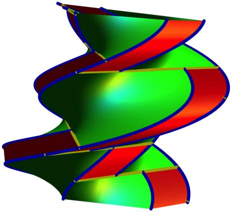
D. Atariah G. Rote M. Wintraecken

Freie Universität Berlin

CGL Workshop, 1.10.2013

- Using Taylor expansion every smooth surface is, locally, approximated by a quadratic patch
- Using Euclidean motions, the quadratic patches can be transformed to graphs of bi-variate polynomials
- So, lets approximate quadratic *graphs*!

$$\{(x, y, z) : z = F(x, y)\}$$



Outline

Introduction

Interpolating Approximation

Non-interpolating Approximation

Vertical Distance

- We are interested, w.l.o.g, in a neighborhood of the origin
- In this case the normal points upwards, and the following can approximate the Hausdorff distance

Definition (Vertical Distance)

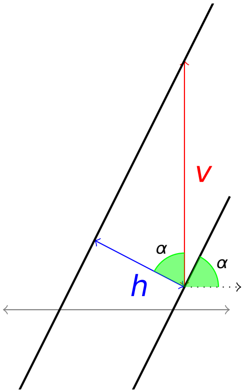
Given two domains $D_1, D_2 \subset \mathbb{R}^2$ and two graphs $f\colon D_1 \to \mathbb{R}$ and $g\colon D_2 \to \mathbb{R}$ then the *vertical distance* is

$$\operatorname{dist}_V(f,g) = \max_{(x,y)\in D_1\cap D_2} |f(x,y) - g(x,y)|.$$

Lemma

Let $A, B \subset \mathbb{R}^3$ be two sets such that their projection to the plane is identical. Then the following holds

$$\mathrm{dist}_H(A, B) \leq \mathrm{dist}_V(A, B)$$



Lemma (Every two points are the same)

For every point $p \in S$, there exists an affine transformation $\mathcal{T}_p : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies the following:

- $\mathcal{T}_p(p) = \vec{0}$
- $\mathcal{T}_p(S) = \tilde{S}$ *which is a quadratic graph given by a polynomial of the form* $\tilde{F}(x, y) = a_1 x^2 + 2a_2 xy + a_3 y^2$
- $\forall q, r \in \mathbb{R}^3$ *which lie on a vertical line we have*

$$|q - r| = |\mathcal{T}_p(q) - \mathcal{T}_p(r)|.$$

Some Properties of V-Dist (cont.)

Lemma

Given two points p, q on a quadratic graph S then

$$\operatorname{dist}_V\left(\ell_{pq}, S\right) = \frac{1}{4}\left|\tilde{F}(p_x - q_x, p_y - q_y)\right|$$

where:

- *ℓ_{pq} is the line segment connecting p and q*
- *$\tilde{F}(x, y)$ is the bi-variate polynomial*
- *V-dist is attained at the midpoint*

For the sake of simplicity, from now on $S = \{ (x, y, z) : z = xy \}$

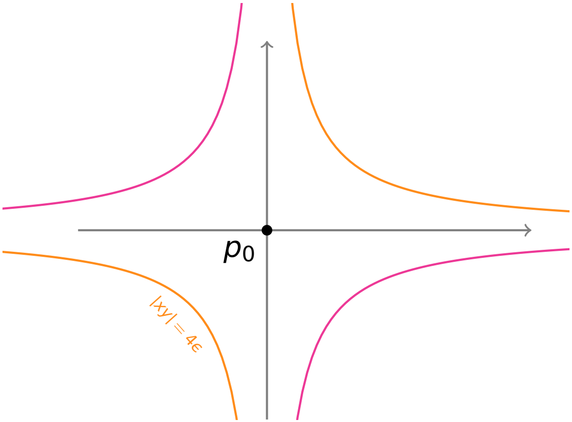
Goal

Find a triangle T with vertices $p_0, p_1, p_2 \in S$ of *maximal area* such that

$$\mathrm{dist}_V(T, S) \leq \epsilon$$

for some $\epsilon > 0$.

Optimize the Area of Planar Triangles



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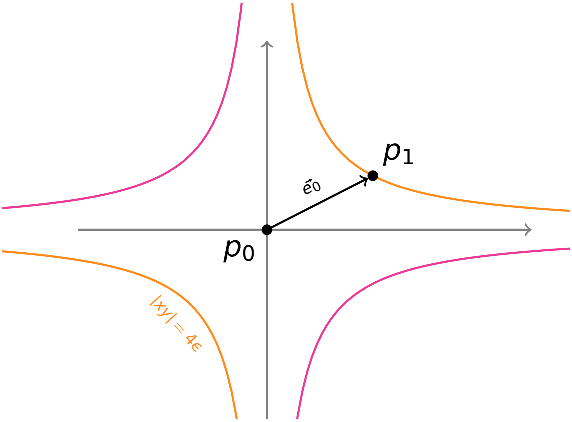

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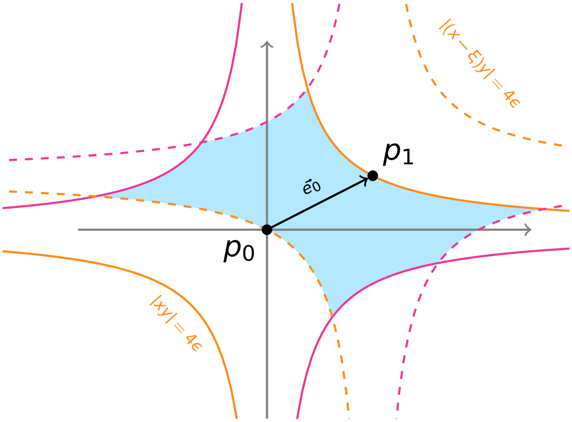

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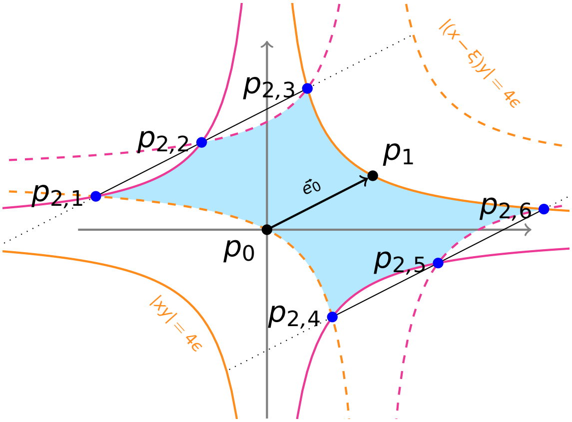

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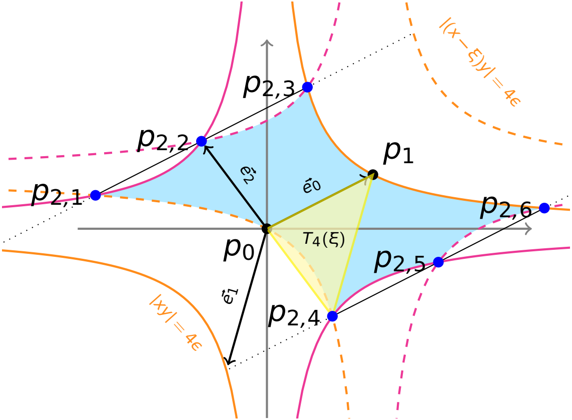
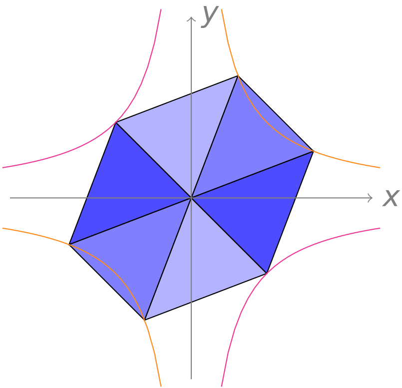


Optimize the Shape of Planar Triangles

Once optimizing the shape of the triangles of maximal area we obtain the following:



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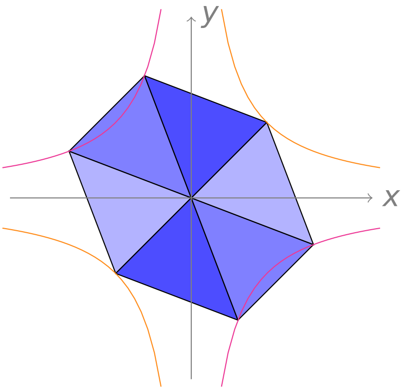
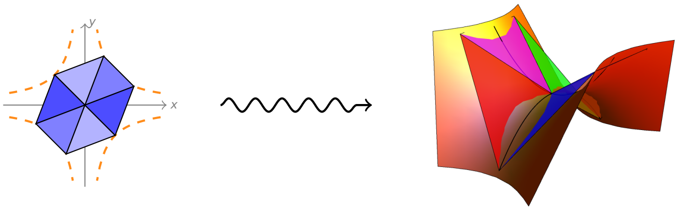
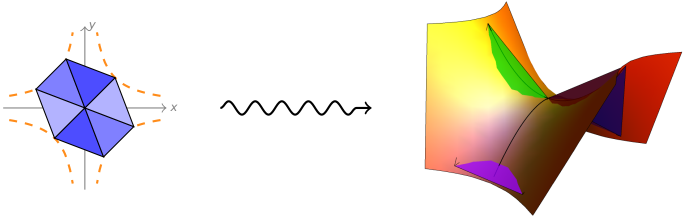


Project the planar triangulation to the surface





Can We Do Better?

What do we have?

Given an $\epsilon > 0$ and a saddle surface S, we can find a family $\mathcal{T}$ of triangles which interpolate the surface and

- have maximal area,
- optimal shape and
- maintain $\text{dist}_V(S, T) = \epsilon$ for all $T \in \mathcal{T}$.

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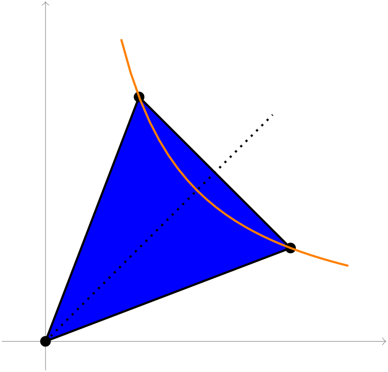
Question...

Can this be improved by allowing *non-interpolating* triangles?

In the Plane

Fact

The area of the (interpolating) optimal triangles in the plane is $2\sqrt{5}\epsilon$.



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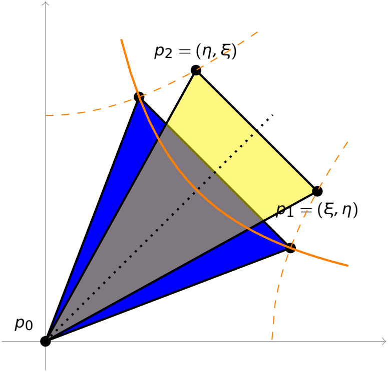

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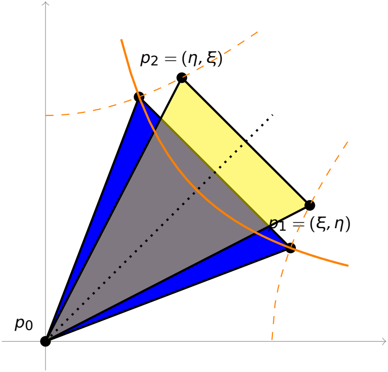

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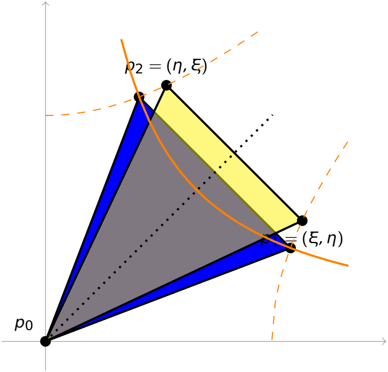

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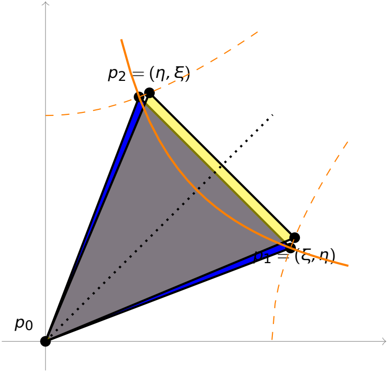

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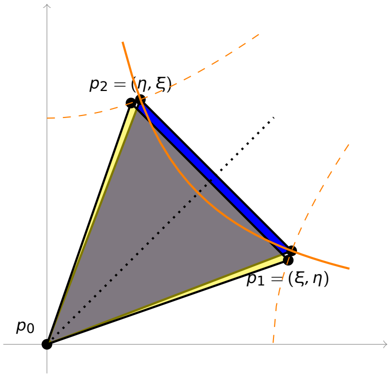

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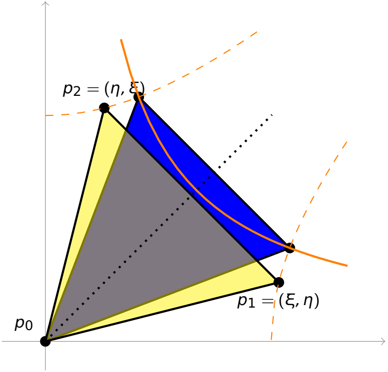
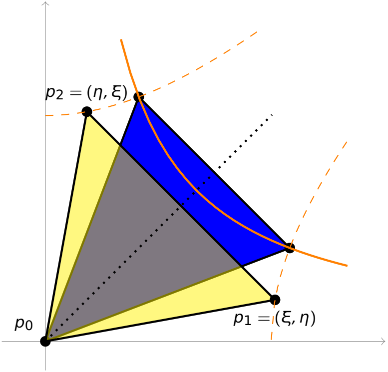


Fact

The area of the (interpolating) optimal triangles in the plane is $2\sqrt{5}\epsilon$.

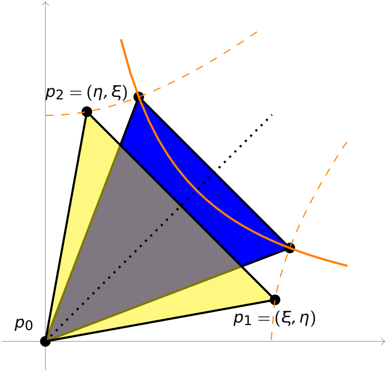
- Obtain one parameter family of area preserving triangles



Fact

The area of the (interpolating) optimal triangles in the plane is $2\sqrt{5}\epsilon$.

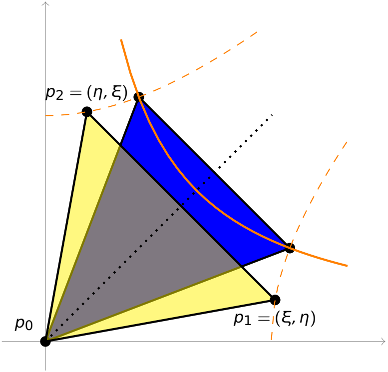
- Obtain one parameter family of area preserving triangles
- How should they be lifted?



- Find vertical lifting $P_i(\xi, \eta)$ of $p_i(\xi, \eta)$ such that

$$\mathrm{dist}_V\,(S, \Delta P(\xi, \eta))$$

will be minimized



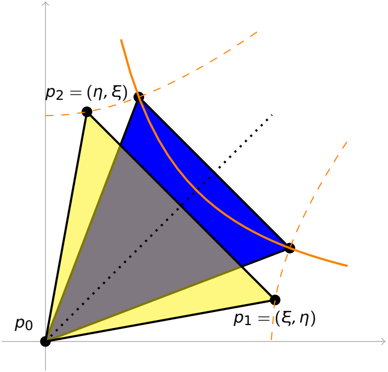
Vertical Perturbed Projection

- Find vertical lifting $P_i(\xi, \eta)$ of $p_i(\xi, \eta)$ such that

$$\mathrm{dist}_V\,(S, \Delta P(\xi, \eta))$$

  will be minimized
- Let $\Delta P_\alpha(\xi, \eta)$ be the projected triangle with vertices on

$$S_\alpha = \left\{ (x, y, z) : z = xy + \alpha \right\}$$



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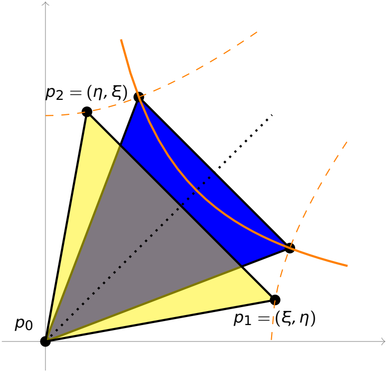
- Vertical distance is attained at midpoints

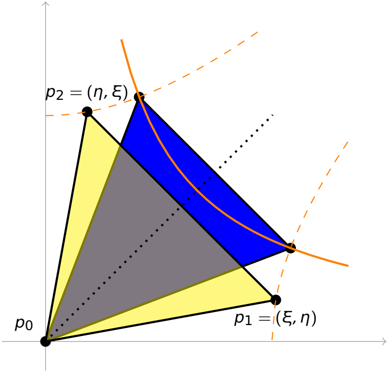


- Vertical distances from edges to S are

$$\frac{\xi\eta}{4} + \alpha > 0$$

$$\frac{1}{4}(\xi - \eta)^2 - \alpha > 0$$

and has to be the same



Vertical Perturbed Projection (Cont.)

- Vertical distances from edges to S are

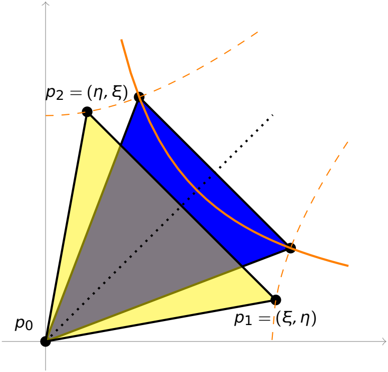
$$\frac{\xi\eta}{4} + \alpha > 0$$

$$\frac{1}{4}(\xi - \eta)^2 - \alpha > 0$$

and has to be the same

- Therefore

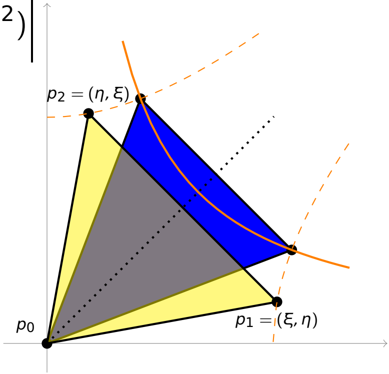
$$\alpha = \frac{1}{8}(\xi^2 - 3\xi\eta + \eta^2)$$



- The vertical distance is

$$\mathrm{dist}_V\,(S, \Delta P_\alpha(\xi)) = \left|\frac{1}{8}(\xi^2 - \xi\eta + \eta^2)\right|$$

and its minimum can be found



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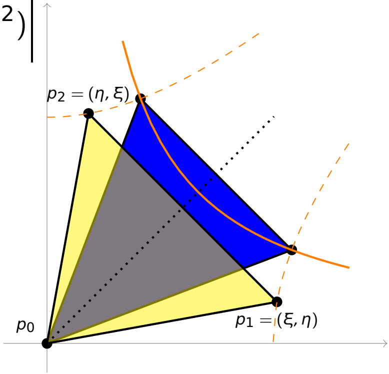
- Min is attained for

$$\xi_0 = \sqrt{2\sqrt{5}\epsilon \frac{2+\sqrt{3}}{\sqrt{3}}}$$



Vertical Perturbed Projection (Cont.)

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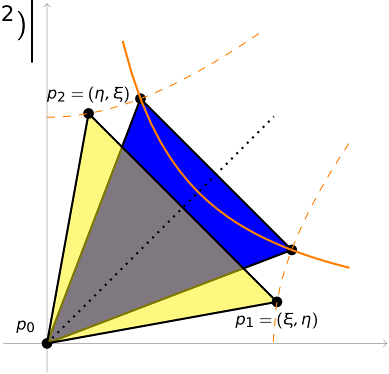
$$\xi_0 = \sqrt{2\sqrt{5}\epsilon \frac{2+\sqrt{3}}{\sqrt{3}}}$$

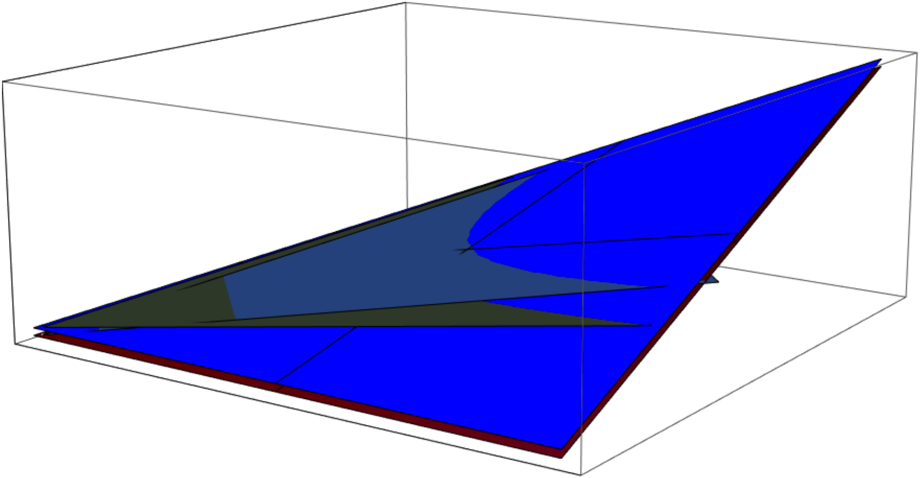
- And in this case

$$\mathrm{dist}_V(S, \Delta P_\alpha(\xi_0)) = \frac{\sqrt{15}}{4}\epsilon \approx 0.968246\epsilon$$



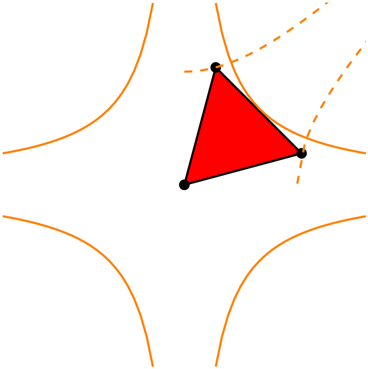
Picture in Space

We can finally plot a non-interpolating optimal triangle which approximates a saddle surface

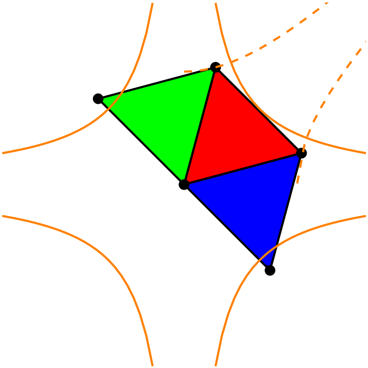


The Planar Super-Optimal Triangle

- Note the tangency property



- Note the tangency property
- Super-optimal triangle is *equilateral!*



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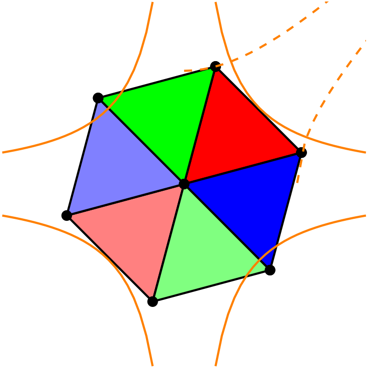

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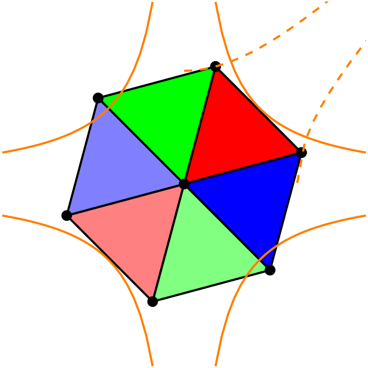

Thank you for your attention!